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SYSTEMATIZATION OF A TYPE SYMMETRIC POLYNOMIALS OF THREE VARIABLES AND SOME APPLICATIONS

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Abstract: Some symmetric polynomials of three variable are represented by elementary symmetric polynomials using the computer program Maple. Such polynomials participate in various problems from the triangle geometry. For this reason some of the results are applied to problems from the triangle geometry. Also, it is considered a problem which could be reduced to such a problem.
Keywords: polynomial, symmetric polynomial, elementary symmetric polynomial, Maple.

A lot of mathematical tasks lead to considerations and examinations of functions of three variables. Algebraic tasks of similar type concern calculating the values of expressions and proofs of identities and inequalities with the participation of three variables. Often, such algebraic tasks are connected with various dependences between the elements of a triangle and the transformation of point coordinates in the plane of a given triangle (complex numbers, barycentric coordinates). In many cases the obtained functions are symmetric polynomials of three variables which are represented by elementary symmetric polynomials for convenience. Thus, we come to the idea of systematizing some frequently used symmetric polynomials and their representation by elementary symmetric polynomials. On the other hand, the representation itself is reduced to the solution of systems linear equations. Very often such systems contain many unknowns, which make them difficult to be solved. In fact, application of convenient computer programs decreases the difficulty. One of the possibilities is to apply the computer program Maple for the purpose and to use it in solving some classical problems.

At the beginning several basic elements of the symmetric polynomial theory are considered.

I. Theoretical background

Each finite sum of type $f\left(x_1, x_2, \ldots, x_n\right) = \sum a_i x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$ is called *symmetric polynomial* $f\left(x_1, x_2, \ldots, x_n\right)$ *of the variables* x_1, x_2, $\ldots$, x_n. If $m = k_1 + k_2 + \cdots + k_n$ is the highest degree of the monomial $a_i x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}$ in the polynomial $f(x_1, x_2, \dots, x_n)$, we say that $f(x_1, x_2, \dots, x_n)$ is *of degree* m. The monomial $a_i x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}$ is called *leading coefficient* of $f(x_1, x_2, \dots, x_n)$.

If each permutation i_1, i_2, $\dots$, i_n of the numbers 1, 2, $\dots$, n satisfies the equality $f(x_{i_1}, x_{i_2}, \dots, x_{i_n}) = f(x_1, x_2, \dots, x_n)$, then the polynomial $f(x_1, x_2, \dots, x_n)$ is called *symmetric*. In other words, if the variables x_1, x_2, $\dots$, x_n are permuted arbitrarily, the polynomial $f(x_1, x_2, \dots, x_n)$ does not change its type.

The symmetric polynomials

$$\sigma_1 = x_1 + x_2 + \dots + x_n,\ \sigma_2 = x_1x_2 + x_1x_3 + \cdots,\ \sigma_3 = x_1x_2x_3 + x_1x_2x_4 + \cdots,\ \sigma_n = x_1x_2 \dots x_n,$$

are called *elementary symmetric polynomials*.

Since the sum, the difference and the product of symmetric polynomials is a symmetric polynomial, it follows that each polynomial of the elementary symmetric polynomials σ_1, σ_2, $\dots$, σ_n is a symmetric polynomial of the variables x_1, x_2, $\dots$, x_n. It turns out, that the reverse assertion is also true and this is the content of the next

Main theorem for symmetric polynomials. *Each symmetric polynomial of the variables* x_1, x_2, $\dots$, x_n *could be represented as a polynomial of the elementary symmetric polynomials* σ_1, σ_2, $\dots$, σ_n.

The main theorem does not say whether the representation is unique. The corresponding answer is given by the next

Theorem for uniqueness. *Each symmetric polynomials of the variables* x_1, x_2, $\dots$, x_n *is represented as a polynomial of the elementary symmetric polynomials* σ_1, σ_2, $\dots$, σ_n *in a unique way.*

It follows from the main theorem and the theorem for uniqueness, that each symmetric polynomial of type $f(x_1, x_2, \dots, x_n) = \sum x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}$ $(k_1 \ge k_2 \ge \dots \ge k_n \ge 0)$ could be represented in a unique way as a sum of monomials of elementary symmetric polynomials, i.e.:

$$f(x_1, x_2, \dots, x_n) = \sigma_1^{k_1-k_2} \sigma_2^{k_2-k_3} \dots \sigma_{n-1}^{k_{n-1}-k_n} \sigma_n^{k_n} +$$
$$+A_1 \sigma_1^{s_1-s_2} \sigma_2^{s_2-s_3} \dots \sigma_{n-1}^{s_{n-1}-s_n} \sigma_n^{s_n} + \dots + A_m \sigma_1^{t_1-t_2} \sigma_2^{t_2-t_3} \dots \sigma_{n-1}^{t_{n-1}-t_n} \sigma_n^{t_n},$$

where the ordered $n-$lets of numbers $(s_1, s_2, \cdots s_n), \dots, (t_1, t_2, \cdots, t_n)$ are all solutions of the equation $x_1 + x_2 + \dots + x_n = k_1 + k_2 + \dots + k_n$, satisfying the conditions $x_1 \ge x_2 \ge \dots \ge x_n \ge 0$.

The solution $(k_1, k_2, \dots, k_n)$, in the case $k_1 \ge k_2 \ge \dots \ge k_n \ge 0$, is called ordered solution of the whole number equation $x_1 + x_2 + \dots + x_n = m$. If $(l_1, l_2, \dots, l_n)$ and $(k_1, k_2, \dots, k_n)$ are two different ordered solutions of the equation $x_1 + x_2 + \dots + x_n = m$ and if there exists such i $(1 \le i \le n-1)$, satisfying the relations $k_1 = l_1$, $k_2 = l_2$, $\dots$, $k_{i-1} = l_{i-1}$, $k_i > l_i$, then we say that the solution $(l_1, l_2, \dots, l_n)$ is lower down the solution $(k_1, k_2, \dots, k_n)$.

II. Systematization of the simple symmetric polynomials of three variables up to degree $n = 10$.

The symmetric polynomials of the variables a, b and c of degree n, which will be under consideration will be of type $f(a,b,c)=\sum a^{k_1}b^{k_2}c^{k_3}$, where $a^{k_1}b^{k_2}c^{k_3}$ will be the leading coefficient of $f(a,b,c)$.

The elementary symmetric polynomial of the variables a, b and c are the following: $\sigma_1=a+b+c$, $\sigma_2=ab+bc+ca$, $\sigma_3=abc$.

The general scheme of presenting a symmetric polynomial of the type under consideration by means of elementary symmetric polynomials could be realized in four stages:

1. Determination of the ordered solutions of the equation $x_1+x_2+x_3=n$, which are lower down the solution (k_1,k_2,k_3).

2. Composition of the polynomial

$$\varphi(\sigma_1,\sigma_2,\sigma_3)=\sigma_1^{k_1-k_2}\sigma_2^{k_2-k_3}\sigma_3^{k_3}+A_1\sigma_1^{s_1-s_2}\sigma_2^{s_2-s_3}\sigma_3^{s_3}+\cdots+A_m\sigma_1^{t_1-t_2}\sigma_2^{t_2-t_3}\sigma_3^{t_3},$$

where $(k_1,k_2,\ldots,k_n)$, $(s_1,s_2,\cdots s_n),\ldots,$ $(t_1,t_2,\cdots,t_n)$ are the consecutive ordered solutions of the equation $x_1+x_2+x_3=n$.

3. Substitution of the variable triplet (a,b,c) in the equality $\varphi(\sigma_1,\sigma_2,\sigma_3)-f(a,b,c)=0$ by m different real number triplets (a_i,b_i,c_i) $(i=1,2\ldots,m)$ and composition of the system

$$\left|\begin{array}{l} f_1(A_1,A_2,\ldots,A_m)=\varphi(\sigma_1,\sigma_2,\sigma_3)-f(a_1,b_1,c_1)=0,\\ f_2(A_1,A_2,\ldots,A_m)=\varphi(\sigma_1,\sigma_2,\sigma_3)-f(a_2,b_2,c_2)=0,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ f_m(A_1,A_2,\ldots,A_m)=\varphi(\sigma_1,\sigma_2,\sigma_3)-f(a_m,b_m,c_m)=0 \end{array}\right.$$

of m linear equations with m unknowns A_1, A_2, $\ldots$, A_m.

4. Solving the system and substituting the constants A_1, A_2, $\ldots$, A_m in the expression for $\varphi(\sigma_1,\sigma_2,\sigma_3)$ from item **2**.

Remark. It could be shown that the number $p_3(n)$ of all ordered solutions of the equation $x_1+x_2+x_3=n$ is determined by means of the next formulae

$$p_3(n)=\begin{cases} \dfrac{n^2}{12}+\dfrac{n}{2}+1, & n\equiv 0(\mathrm{mod}\,6),\\ \dfrac{n^2}{12}+\left[\dfrac{n}{2}\right]+\dfrac{11}{12}, & n\equiv 1(\mathrm{mod}\,6),\\ \dfrac{n^2}{12}+\dfrac{n}{2}+\dfrac{2}{3}, & n\equiv 2(\mathrm{mod}\,6),\\ \dfrac{n^2}{12}+\left[\dfrac{n}{2}\right]+\dfrac{5}{4}, & n\equiv 3(\mathrm{mod}\,6),\\ \dfrac{n^2}{12}+\dfrac{n}{2}+\dfrac{2}{3}, & n\equiv 4(\mathrm{mod}\,6),\\ \dfrac{n^2}{12}+\left[\dfrac{n}{2}\right]+\dfrac{11}{12}, & n\equiv 5(\mathrm{mod}\,6). \end{cases}$$

The described scheme for the polynomial $a^4b+ab^4+b^4c+bc^4+c^4a+ca^4$, realized by Maple, appears in the following way:

(4,1,0):

$$a^4b+a\cdot b^4+b^4\cdot c+b\cdot c^4+c^4\cdot a+c\cdot a^4=\sigma 1^3\,\sigma 2-3\,\sigma 1\,\sigma 2^2-\sigma 1^2\,\sigma 3+5\,\sigma 2\,\sigma 3;$$

> $restart;$

> $a:=1;\ b:=1;\ c:=1;$

$a:=1$

$b:=1$

$c:=1$

>

> $s1:=a+b+c;\ s2:=a\cdot b+b\cdot c+c\cdot a;\ s3:=a\cdot b\cdot c;$

$s1:=3$

$s2:=3$

$s3:=1$

> $f:=a^4b+a\cdot b^4+b^4\cdot c+b\cdot c^4+c^4\cdot a+c\cdot a^4;$

$f:=6$

> $f0:=s1^{4-1}\cdot s2^{1-0}\cdot s3^0+A1\cdot s1^{3-2}\cdot s2^{2-0}\cdot s3^0+A2\cdot s1^{3-1}\cdot s2^{1-1}\cdot s3^1+A3\cdot s1^{2-2}\cdot s2^{2-1}\cdot s3^1;$

$f0:=81+27\,A1+9\,A2+3\,A3$

> $f1:=f0-f;$

$f1:=75+27\,A1+9\,A2+3\,A3$

> $a:=a+1;\ b:=b+2;\ c:=c+3;$

$a:=2$

$b:=3$

$c:=4$

> $s1:=a+b+c;\ s2:=a\cdot b+b\cdot c+c\cdot a;\ s3:=a\cdot b\cdot c;$

$s1:=9$

$s2:=26$

$s3:=24$

> $f:=a^4b+a\cdot b^4+b^4\cdot c+b\cdot c^4+c^4\cdot a+c\cdot a^4;$

$f:=1878$

> $f0:=s1^{4-1}\cdot s2^{1-0}\cdot s3^0+A1\cdot s1^{3-2}\cdot s2^{2-0}\cdot s3^0+A2\cdot s1^{3-1}\cdot s2^{1-1}\cdot s3^1+A3\cdot s1^{2-2}\cdot s2^{2-1}\cdot s3^1;$

$f0:=18954+6084\,A1+1944\,A2+624\,A3$

>

$f0:=6084+1944\,A1+624\,A2$

> $f2:=f0-f;$

$f2:=17076+6084\,A1+1944\,A2+624\,A3$

> $a:=a+1;\ b:=b+2;\ c:=c+3;$

$$a := 3$$
$$b := 5$$
$$c := 7$$

> $s1 := a + b + c; s2 := a \cdot b + b \cdot c + c \cdot a; s3 := a \cdot b \cdot c;$

$$s1 := 15$$
$$s2 := 71$$
$$s3 := 105$$

> $f := a^4 b + a \cdot b^4 + b^4 \cdot c + b \cdot c^4 + c^4 \cdot a + c \cdot a^4;$

$$f := 26430$$

> $f0 := s1^{4-1} \cdot s2^{1-0} \cdot s3^0 + A1 \cdot s1^{3-2} \cdot s2^{2-0} \cdot s3^0 + A2 \cdot s1^{3-1} \cdot s2^{1-1} \cdot s3^1 + A3 \cdot s1^{2-2} \cdot s2^{2-1} \cdot s3^1;$

$$f0 := 239625 + 75615\,A1 + 23625\,A2 + 7455\,A3$$

> $f3 := f0 - f;$

$$f3 := 213195 + 75615\,A1 + 23625\,A2 + 7455\,A3$$

>

>

> $solve(\{f1, f2, f3\}, \{A1, A2, A3\});$

$$\{A1 = -3, A2 = -1, A3 = 5\}$$

> $restart;$

> $f0 := s1^{4-1} \cdot s2^{1-0} \cdot s3^0 + A1 \cdot s1^{3-2} \cdot s2^{2-0} \cdot s3^0 + A2 \cdot s1^{3-1} \cdot s2^{1-1} \cdot s3^1 + A3 \cdot s1^{2-2} \cdot s2^{2-1} \cdot s3^1;$

$$f0 := s1^3\, s2 + A1\, s1\, s2^2 + A2\, s1^2\, s3 + A3\, s2\, s3$$

> $s1 := sigma1; s2 := sigma2; s3 := sigma3;$

$$s1 := \sigma1$$
$$s2 := \sigma2$$
$$s3 := \sigma3$$

> $A1 := -3; A2 := -1; A3 := 5;$

$$A1 := -3$$
$$A2 := -1$$
$$A3 := 5$$

> $f0;$

$$\sigma1^3\, \sigma2 - 3\, \sigma1\, \sigma2^2 - \sigma1^2\, \sigma3 + 5\, \sigma2\, \sigma3$$

> $restart;$

> $A1 := -3; A2 := -1; A3 := 5;$

$$A1 := -3$$
$$A2 := -1$$
$$A3 := 5$$

> $s1 := a + b + c; s2 := a \cdot b + b \cdot c + c \cdot a; s3 := a \cdot b \cdot c;$

$$s1 := a + b + c$$
$$s2 := a\, b + b\, c + c\, a$$
$$s3 := a\, b\, c$$

> $f0 := s1^{4-1} \cdot s2^{1-0} \cdot s3^{0} + A1 \cdot s1^{3-2} \cdot s2^{2-0} \cdot s3^{0} + A2 \cdot s1^{3-1} \cdot s2^{1-1} \cdot s3^{1} + A3 \cdot s1^{2-2} \cdot s2^{2-1} \cdot s3^{1};$

$$f0 := (a+b+c)^3 (ab+bc+ca) - 3(a+b+c)(ab+bc+ca)^2 - (a+b+c)^2 abc + 5(ab+bc+ca)abc$$

> $f := a^4 b + a \cdot b^4 + b^4 \cdot c + b \cdot c^4 + c^4 \cdot a + c \cdot a^4;$

$$f := a^4 b + a b^4 + b^4 c + b c^4 + c^4 a + c a^4$$

> $f - f0;$

$$a^4 b + a b^4 + b^4 c + b c^4 + c^4 a + c a^4 - (a+b+c)^3 (ab+bc+ca) + 3(a+b+c)(ab+bc+ca)^2 + (a+b+c)^2 abc - 5(ab+bc+ca)abc$$

> *factor*(**(14.26)**)

$$0$$

In a similar way following the described scheme we obtain the next systematization of polynomials of degree n, when $1 \le n \le 10$.

$n=1$		
Ordered solutions of the equation $x_1+x_2+x_3=1$	$f(a,b,c)$	$\varphi(\sigma_1,\sigma_2,\sigma_3)$
$(1,0,0)$	$a+b+c$	σ_1

$n=2$		
Ordered solutions of the equation $x_1+x_2+x_3=2$	$f(a,b,c)$	$\varphi(\sigma_1,\sigma_2,\sigma_3)$
$(1,1,0)$	$ab+bc+ca$	σ_2
$(2,0,0)$	$a^2+b^2+c^2$	$\sigma_1^2-2\sigma_1\sigma_2$

$n=3$		
Ordered solutions of the equation $x_1+x_2+x_3=3$	$f(a,b,c)$	$\varphi(\sigma_1,\sigma_2,\sigma_3)$
$(1,1,1)$	abc	σ_3
$(2,1,0)$	$a^2b+ab^2+b^2c+bc^2+c^2a+ca^2$	$\sigma_1\sigma_2-3\sigma_3$
$(3,0,0)$	$a^3+b^3+c^3$	$\sigma_1^3-3\sigma_1\sigma_2+3\sigma_3$

$n=4$		
Ordered solutions	$f(a,b,c)$	$\varphi(\sigma_1,\sigma_2,\sigma_3)$

of the equation $x_1+x_2+x_3=4$		
$(2,2,0)$	$a^2b^2+b^2c^2+c^2a^2$	$\sigma_2^2-2\sigma_1\sigma_3$
$(2,1,1)$	$a^2bc+ab^2c+abc^2$	$\sigma_1\sigma_3$
$(3,1,0)$	$a^3b+ab^3+b^3c+bc^3+c^3a+ca^3$	$\sigma_1^2\sigma_2-2\sigma_2^2-\sigma_1\sigma_3$
$(4,0,0)$	$a^4+b^4+c^4$	$\sigma_1^4-4\sigma_1^2\sigma_2+2\sigma_2^2+4\sigma_1\sigma_3$

$n=5$		
Ordered solutions of the equation $x_1+x_2+x_3=5$	$f(a,b,c)$	$\varphi(\sigma_1,\sigma_2,\sigma_3)$
$(2,2,1)$	$a^2b^2c+a^2bc^2+ab^2c^2$	$\sigma_2\sigma_3$
$(3,2,0)$	$a^3b^2+a^2b^3+b^3c^2+b^2c^3+c^3a^2+c^2a^3$	$\sigma_1\sigma_2^2-2\sigma_1^2\sigma_3-\sigma_2\sigma_3$
$(3,1,1)$	$a^3bc+ab^3c+abc^3$	$\sigma_2\sigma_3$
$(4,1,0)$	$a^4b+ab^4+b^4c+bc^4+c^4a+ca^4$	$\sigma_1^3\sigma_2-3\sigma_1\sigma_2^2-\sigma_1^2\sigma_3+5\sigma_2\sigma_3$
$(5,0,0)$	$a^5+b^5+c^5$	$\sigma_1^5-5\sigma_1^3\sigma_2+5\sigma_1\sigma_2^2+5\sigma_1^2\sigma_3-5\sigma_2\sigma_3$

$n=6$		
Ordered solutions of the equation $x_1+x_2+x_3=6$	$f(a,b,c)$	$\varphi(\sigma_1,\sigma_2,\sigma_3)$
$(2,2,2)$	$a^2b^2c^2$	σ_3^2
$(3,2,1)$	$a^3b^2c+a^2b^3c+ab^3c^2+$ $+ab^2c^3+bc^3a^2+bc^2a^3$	$\sigma_3(\sigma_1\sigma_2-3\sigma_3)$
$(3,3,0)$	$a^3b^3+b^3c^3+c^3a^3$	$\sigma_2^3-3\sigma_1\sigma_2\sigma_3+3\sigma_3^2$
$(4,2,0)$	$a^4b^2+a^2b^4+b^4c^2+$ $+b^2c^4+c^4a^2+c^2a^4$	$\sigma_1^2\sigma_2^2-2\sigma_1^3\sigma_3-$ $-2\sigma_2^3+4\sigma_1\sigma_2\sigma_3-3\sigma_3^2$
$(4,1,1)$	$a^4bc+ab^4c+abc^4$	$\sigma_3(\sigma_1^3-3\sigma_1\sigma_2+3\sigma_3)$
$(5,1,0)$	$a^5b+ab^5+b^5c+bc^5+c^5a+ca^5$	$\sigma_1^4\sigma_2-4\sigma_1^2\sigma_2^2-\sigma_1^3\sigma_3+$ $+2\sigma_2^3+7\sigma_1\sigma_2\sigma_3-3\sigma_3^2$
$(6,0,0)$	$a^6+b^6+c^6$	$\sigma_1^6-6\sigma_1^4\sigma_2+9\sigma_1^2\sigma_2^2+6\sigma_1^3\sigma_3-$ $-2\sigma_2^3-12\sigma_1\sigma_2\sigma_3+3\sigma_3^2$

$n=7$		
Ordered solutions of the equation $x_1+x_2+x_3=7$	$f(a,b,c)$	$\varphi(\sigma_1,\sigma_2,\sigma_3)$

$(3,3,1)$	$a^3b^3c+a^3bc^3+ab^3c^3$	$\sigma_3\left(\sigma_2^2-2\sigma_1\sigma_3\right)$
$(3,2,2)$	$a^3b^2c^2+a^2b^3c^2+a^2b^3c^2$	$\sigma_3^2\sigma_2$
$(4,3,0)$	$a^4b^3+a^3b^4+b^4c^3+$ $+b^3c^4+c^4a^3+c^3a^4$	$\sigma_1\sigma_2^3-3\sigma_1^2\sigma_2\sigma_3-\sigma_2^2\sigma_3+5\sigma_1\sigma_3^2$
$(4,2,1)$	$a^4b^2c+a^2b^4c+ab^4c^2+$ $+ab^2c^4+bc^4a^2+bc^2a^4$	$\sigma_3\left(\sigma_1^2\sigma_2-2\sigma_2^2-\sigma_1\sigma_3\right)$
$(5,2,0)$	$a^5b^2+a^2b^5+b^5c^2+$ $+b^2c^5+c^5a^2+c^2a^5$	$\sigma_1^3\sigma_2^2-2\sigma_1^4\sigma_3-3\sigma_1\sigma_2^3+$ $+6\sigma_1^2\sigma_2\sigma_3+3\sigma_2^2\sigma_3-7\sigma_1\sigma_3^2$
$(5,1,1)$	$a^5bc+ab^5c+abc^5$	$\sigma_3\left(\sigma_1^4-4\sigma_1^2\sigma_2+2\sigma_2^2+4\sigma_1\sigma_3\right)$
$(6,1,0)$	$a^6b+ab^6+b^6c+bc^6+c^6a+ca^6$	$\sigma_1^5\sigma_2-5\sigma_1^3\sigma_2^2-\sigma_1^4\sigma_3+5\sigma_1\sigma_2^3+$ $+9\sigma_1^2\sigma_2\sigma_3-7\sigma_2^2\sigma_3-4\sigma_1\sigma_3^2$
$(7,0,0)$	$a^7+b^7+c^7$	$\sigma_1^7-7\sigma_1^5\sigma_2+14\sigma_1^3\sigma_2^2+7\sigma_1^4\sigma_3-$ $-7\sigma_1\sigma_2^3-21\sigma_1^2\sigma_2\sigma_3+7\sigma_2^2\sigma_3+7\sigma_1\sigma_3^2$

$n=8$		
Ordered solutions of the equation $x_1+x_2+x_3=8$	$f(a,b,c)$	$\varphi(\sigma_1,\sigma_2,\sigma_3)$
$(3,3,2)$	$a^3b^3c^2+a^3b^2c^3+a^2b^3c^3$	$\sigma_2\sigma_3^2$
$(4,4,0)$	$a^4b^4+b^4c^4+c^4a^4$	$\sigma_2^4-4\sigma_1\sigma_2^2\sigma_3+2\sigma_1^2\sigma_3^2+4\sigma_2\sigma_3^2$
$(4,3,1)$	$a^4b^3c+a^3b^4c+ab^4c^3+$ $+ab^3c^4+bc^4a^3+bc^3a^4$	$\sigma_3\left(\sigma_1\sigma_2^2-2\sigma_1^2\sigma_3-\sigma_2\sigma_3\right)$
$(4,2,2)$	$a^4b^2c^2+a^2b^4c^2+a^2b^4c^2$	$\sigma_3^2\left(\sigma_1^2-2\sigma_2\right)$
$(5,3,0)$	$a^5b^3+a^3b^5+b^5c^3+$ $+b^3c^5+c^5a^3+c^3a^5$	$\sigma_1^2\sigma_2^3-3\sigma_1^3\sigma_2\sigma_3-2\sigma_2^4+$ $+6\sigma_1\sigma_2^2\sigma_3+3\sigma_1^2\sigma_3^2-7\sigma_2\sigma_3^2$
$(5,2,1)$	$a^5b^2c+a^2b^5c+ab^5c^2+$ $+ab^2c^5+bc^5a^2+bc^2a^5$	$\sigma_3\left(\sigma_1^3\sigma_2-3\sigma_1\sigma_2^2-\sigma_1^2\sigma_3+5\sigma_2\sigma_3\right)$
$(6,2,0)$	$a^6b^2+a^2b^6+b^6c^2+$ $+b^2c^6+c^6a^2+c^2a^6$	$\sigma_1^4\sigma_2^2-2\sigma_1^5\sigma_3-4\sigma_1^2\sigma_2^3+$ $+8\sigma_1^3\sigma_2\sigma_3+2\sigma_1^4-9\sigma_1^2\sigma_3^2+2\sigma_2\sigma_3^2$
$(6,1,1)$	$a^6bc+ab^6c+abc^6$	$\sigma_3\left(\sigma_1^5-5\sigma_1^3\sigma_2+5\sigma_1\sigma_2^2+5\sigma_1^2\sigma_3-5\sigma_2\sigma_3\right)$
$(7,1,0)$	$a^7b+ab^7+b^7c+bc^7+c^7a+ca^7$	$\sigma_1^6\sigma_2-6\sigma_1^4\sigma_2^2-\sigma_1^5\sigma_3+$ $+9\sigma_1^2\sigma_2^3+11\sigma_1^3\sigma_2\sigma_3-2\sigma_2^4-$ $-17\sigma_1\sigma_2^2\sigma_3-5\sigma_1^2\sigma_3^2+8\sigma_2\sigma_3^2$
$(8,0,0)$	$a^8+b^8+c^8$	$\sigma_1^8-8\sigma_1^6\sigma_2+20\sigma_1^4\sigma_2^2+8\sigma_1^5\sigma_3-$ $-16\sigma_1^2\sigma_2^3-32\sigma_1^3\sigma_2\sigma_3+2\sigma_2^4+$ $+24\sigma_1\sigma_2^2\sigma_3+12\sigma_1^2\sigma_3^2-8\sigma_2\sigma_3^2$

$n=9$		
Ordered solutions of the equation $x_1+x_2+x_3=9$	$f(a,b,c)$	$\varphi(\sigma_1,\sigma_2,\sigma_3)$
$(3,3,3)$	$a^3b^3c^3$	σ_3^3
$(4,4,1)$	$a^4b^4c+b^4c^4a+c^4a^4b$	$\sigma_3\left(\sigma_2^3-3\sigma_1\sigma_2\sigma_3+3\sigma_3^2\right)$
$(4,3,2)$	$a^4b^3c^2+a^3b^4c^2+a^2b^4c^3+$ $+a^2b^3c^4+b^2c^4a^3+b^2c^3a^4$	$\sigma_3^2\left(\sigma_1\sigma_2-3\sigma_3\right)$
$(5,4,0)$	$a^5b^4+a^4b^5+b^5c^4+$ $+b^4c^5+c^5a^4+c^4a^5$	$\sigma_1\sigma_2^4-4\sigma_1^2\sigma_2^2\sigma_3+2\sigma_1^3\sigma_3^2-$ $-\sigma_2^3\sigma_3+7\sigma_1\sigma_2\sigma_3^2-3\sigma_3^3$
$(5,3,1)$	$a^5b^3c+a^3b^5c+ab^5c^3+$ $+ab^3c^5+bc^5a^3+bc^3a^5$	$\sigma_3\left(\sigma_1^2\sigma_2^2-2\sigma_1^3\sigma_3-\right.$ $\left.-2\sigma_2^3+4\sigma_1\sigma_2\sigma_3-3\sigma_3^2\right)$
$(5,2,2)$	$a^5b^2c^2+a^2b^5c^2+a^2b^2c^5$	$\sigma_3^2\left(\sigma_1^3-3\sigma_1\sigma_2+3\sigma_3\right)$
$(6,3,0)$	$a^6b^3+a^3b^6+b^6c^3+$ $+b^3c^6+c^6a^3+c^3a^6$	$\sigma_1^3\sigma_2^3-3\sigma_1^4\sigma_2\sigma_3-3\sigma_1\sigma_2^4+9\sigma_1^2\sigma_2^2\sigma_3+$ $+3\sigma_1^3\sigma_3^2+3\sigma_2^3\sigma_3-18\sigma_1\sigma_2\sigma_3^2+6\sigma_3^3$
$(6,2,1)$	$a^6b^2c+a^2b^6c+ab^6c^2+$ $+ab^2c^6+bc^6a^2+bc^2a^6$	$\sigma_3\left(\sigma_1^4\sigma_2-4\sigma_1^2\sigma_2^2-\sigma_1^3\sigma_3+\right.$ $\left.+2\sigma_2^3+7\sigma_1\sigma_2\sigma_3-3\sigma_3^2\right)$
$(7,2,0)$	$a^7b^2+a^2b^7+b^7c^2+$ $+b^2c^7+c^7a^2+c^2a^7$	$\sigma_1^5\sigma_2^2-2\sigma_1^6\sigma_3-5\sigma_1^3\sigma_2^3+$ $+10\sigma_1^4\sigma_2\sigma_3+5\sigma_1\sigma_2^4-5\sigma_1^2\sigma_2^2\sigma_3-$ $-11\sigma_1^3\sigma_3^2-5\sigma_2^3\sigma_3+13\sigma_1\sigma_2\sigma_3^2-3\sigma_3^3$
$(7,1,1)$	$a^7bc+ab^7c+abc^7$	$\sigma_3\left(\sigma_1^6-6\sigma_1^4\sigma_2+9\sigma_1^2\sigma_2^2+\right.$ $\left.+6\sigma_1^3\sigma_3-2\sigma_2^3-12\sigma_1\sigma_2\sigma_3+3\sigma_3^2\right)$
$(8,1,0)$	$a^8b+ab^8+b^8c+bc^8+c^8a+ca^8$	$\sigma_1^7\sigma_2-7\sigma_1^5\sigma_2^2-\sigma_1^6\sigma_3+14\sigma_1^3\sigma_2^3+$ $+13\sigma_1^4\sigma_2\sigma_3-7\sigma_1\sigma_2^4-30\sigma_1^2\sigma_2^2\sigma_3-$ $-6\sigma_1^3\sigma_3^2+9\sigma_2^3+19\sigma_1\sigma_2\sigma_3^2-3\sigma_3^3$
$(9,0,0)$	$a^9+b^9+c^9$	$\sigma_1^9-9\sigma_1^7\sigma_2+27\sigma_1^5\sigma_2^2+9\sigma_1^6\sigma_3-$ $-30\sigma_1^3\sigma_2^3-45\sigma_1^4\sigma_2\sigma_3+9\sigma_1\sigma_2^4+$ $+54\sigma_1^2\sigma_2^2\sigma_3+18\sigma_1^3\sigma_3^2-9\sigma_2^3\sigma_3-$ $-27\sigma_1\sigma_2\sigma_3^3+3\sigma_3^3$

$n=10$		
Ordered solutions of the equation $x_1+x_2+x_3=10$	$f(a,b,c)$	$\varphi(\sigma_1,\sigma_2,\sigma_3)$

$(4,4,2)$	$a^4b^4c^2+b^4c^4a^2+c^4a^4b^2$	$\sigma_3^2\left(\sigma_2^2-2\sigma_1\sigma_3\right)$
$(4,3,3)$	$a^4b^3c^3+b^4c^3a^3+c^4a^3b^3$	$\sigma_1\sigma_3^3$
$(5,5,0)$	$a^5b^5+b^5c^5+c^5a^5$	$\sigma_2^5-5\sigma_1\sigma_2^3\sigma_3+5\sigma_1^2\sigma_3^2+5\sigma_2^2\sigma_3^2-5\sigma_1\sigma_3^3$
$(5,4,1)$	$a^5b^4c+a^4b^5c+ab^5c^4+$ $+ab^4c^5+bc^5a^4+bc^4a^5$	$\sigma_3\left(\sigma_1\sigma_2^3-3\sigma_1^2\sigma_2\sigma_3-\sigma_2^2\sigma_3+5\sigma_1\sigma_3^2\right)$
$(5,3,2)$	$a^5b^3c^2+a^3b^5c^2+a^2b^5c^3+$ $+a^2b^3c^5+b^2c^5a^3+b^2c^3a^5$	$\sigma_3^2\left(\sigma_1^2\sigma_2-2\sigma_2^2-\sigma_1\sigma_3\right)$
$(6,4,0)$	$a^6b^4+a^4b^6+b^6c^4+$ $+b^4c^6+c^6a^4+c^4a^6$	$\sigma_1^2\sigma_2^4-4\sigma_1^3\sigma_2^2\sigma_3+2\sigma_1^4\sigma_3^2-2\sigma_2^5+$ $+8\sigma_1\sigma_2^3\sigma_3-9\sigma_2^2\sigma_3^2+2\sigma_1\sigma_3^3$
$(6,3,1)$	$a^6b^3c+a^3b^6c+ab^6c^3+$ $+ab^3c^6+bc^6a^3+bc^3a^6$	$\sigma_3\left(\sigma_1^3\sigma_2^2-2\sigma_1^4\sigma_3-3\sigma_1\sigma_2^3+\right.$ $\left.+6\sigma_1^2\sigma_2\sigma_3+3\sigma_2^2\sigma_3-7\sigma_1\sigma_3^2\right)$
$(6,2,2)$	$a^6b^2c^2+a^2b^6c^2+a^2b^2c^6$	$\sigma_3^2\left(\sigma_1^4-4\sigma_1^2\sigma_2+2\sigma_2^2+4\sigma_1\sigma_3\right)$
$(7,3,0)$	$a^7b^3+a^3b^7+b^7c^3+$ $+b^3c^7+c^7a^3+c^3a^7$	$\sigma_1^4\sigma_2^3-3\sigma_1^5\sigma_2\sigma_3-4\sigma_1^2\sigma_2^4+12\sigma_1^3\sigma_2^2\sigma_3+$ $+3\sigma_1^4\sigma_3^2+2\sigma_2^5-2\sigma_1\sigma_2^3\sigma_3-24\sigma_1^2\sigma_2\sigma_3^2+$ $+6\sigma_2^2\sigma_3^2+11\sigma_1\sigma_3^3$
$(7,2,1)$	$a^7b^2c+a^2b^7c+ab^7c^2+$ $+ab^2c^7+bc^7a^2+bc^2a^7$	$\sigma_3\left(\sigma_1^5\sigma_2-5\sigma_1^3\sigma_2^2-\sigma_1^4\sigma_3+5\sigma_1\sigma_2^3+\right.$ $\left.+9\sigma_1^2\sigma_2\sigma_3-7\sigma_2^2\sigma_3-4\sigma_1\sigma_3^2\right)$
$(8,2,0)$	$a^8b^2+a^2b^8+b^8c^2+$ $+b^2c^8+c^8a^2+c^2a^8$	$\sigma_1^6\sigma_2^2-2\sigma_1^7\sigma_3-6\sigma_1^4\sigma_2^3+12\sigma_1^5\sigma_2\sigma_3+$ $+9\sigma_1^2\sigma_2^4-12\sigma_1^3\sigma_2^2\sigma_3-13\sigma_1^4\sigma_3^2-$ $-2\sigma_2^5-8\sigma_1\sigma_2^3\sigma_3+28\sigma_1^2\sigma_2\sigma_3^2+$ $+\sigma_2^2\sigma_3^2-10\sigma_1\sigma_3^3$
$(8,1,1)$	$a^8bc+ab^8c+abc^8$	$\sigma_3\left(\sigma_1^7-7\sigma_1^5\sigma_2+14\sigma_1^3\sigma_2^2+7\sigma_1^4\sigma_3-\right.$ $\left.-7\sigma_1\sigma_2^3-21\sigma_1^2\sigma_2\sigma_3+7\sigma_2^2\sigma_3+7\sigma_1\sigma_3^2\right)$
$(9,1,0)$	$a^9b+ab^9+b^9c+bc^9+c^9a+ca^9$	$\sigma_1^8\sigma_2-8\sigma_1^6\sigma_2^2-\sigma_1^7\sigma_3+20\sigma_1^4\sigma_2^3+$ $+15\sigma_1^5\sigma_2\sigma_3-16\sigma_1^2\sigma_2^4-46\sigma_1^3\sigma_2^2\sigma_3-$ $-7\sigma_1^4\sigma_3^2+2\sigma_2^5+31\sigma_1\sigma_2^3\sigma_3+$ $+33\sigma_1^2\sigma_2\sigma_3^2-15\sigma_2^2\sigma_3^2-7\sigma_1\sigma_3^3$
$(10,0,0)$	$a^{10}+b^{10}+c^{10}$	$\sigma_1^{10}-10\sigma_1^8\sigma_2+35\sigma_1^6\sigma_2^2+10\sigma_1^7\sigma_3-$ $-50\sigma_1^4\sigma_2^3-60\sigma_1^5\sigma_2\sigma_3+25\sigma_1^2\sigma_2^4+$ $+100\sigma_1^3\sigma_2^2\sigma_3+25\sigma_1^4\sigma_3^2-2\sigma_2^5-$ $-40\sigma_1\sigma_2^3\sigma_3-60\sigma_1^2\sigma_2\sigma_3^2+$ $+15\sigma_2^2\sigma_3^2+10\sigma_1\sigma_3^3$

III. Some applications

We solve several problems as application of some of the obtained results.

Problem 1. Let $a+b+c=1$, $a^2+b^2+c^2=2$ and $a^3+b^3+c^3=3$. If

$$S(5,3)=a^5b^3+a^3b^5+b^5c^3+b^3c^5+c^5a^3+c^3a^5,$$

$$S(6,2)=a^6b^2+a^2b^6+b^6c^2+b^2c^6+c^6a^2+c^2a^6,$$

prove that $|S(5,3)|>|S(6,2)|$.

Solution. It follows from the condition and from two of the derived formulae, that $\sigma_1=1$, $\sigma_2=-\frac{1}{2}$ and $\sigma_3=\frac{1}{6}$. Applying once again two of the formulae in the implemented table, we obtain $S(5,3)=\frac{31}{72}$, $S(6,2)=-\frac{29}{72}$. Consequently, $|S(5,3)|>|S(6,2)|$.

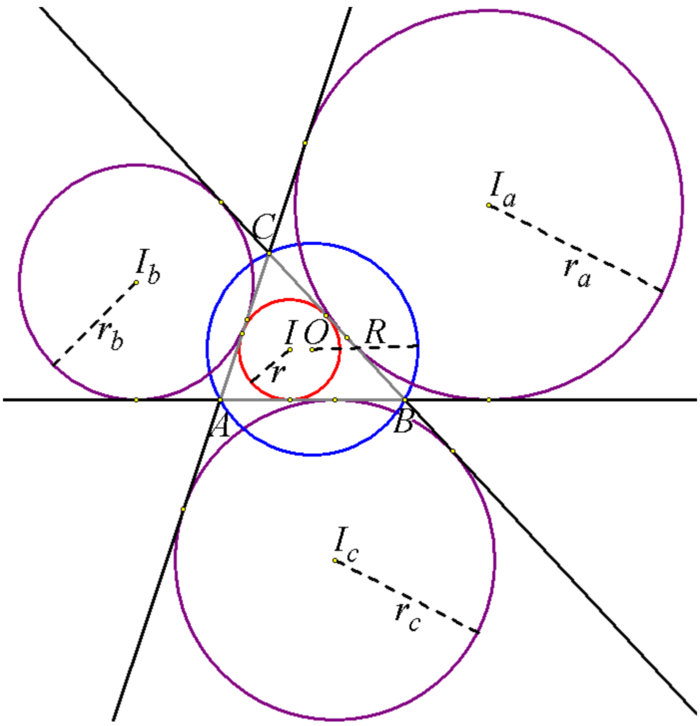
Remark. It follows from Vieta formulae, that the numbers a, b and c are roots of the cubic equation $x^3-x^2-\frac{1}{2}x-\frac{1}{6}=0$. The roots are expressed by the formulae:

$$\frac{A^2+2A+10}{6A},\ \frac{-\left(A^2-4A+10\right)+i\left(A^2-10\right)\sqrt{3}}{12A},\ \frac{-\left(A^2-4A+10\right)-i\left(A^2-10\right)\sqrt{3}}{12A},$$

where $A=\sqrt[3]{44+6\sqrt{26}}$. Such expressions are not suitable for calculation of the above sums.

Problem 2. Triangle ABC with perimeter $2p$ is inscribed in a circle with radius R and is circumscribed with respect to a circle with radius r. If r_a, r_b and r_c are the radii of the excircles of ΔABC, prove the inequalities

$$\frac{p^2r\left(256r^2-37R^2\right)}{2}\le r_a^3r_b^2+r_a^2r_b^3+r_b^3r_c^2+r_b^2r_c^3+r_c^3r_a^2+r_c^2r_a^3\le 2p^2\left(8R^3-37r^3\right).$$



Solution. The following formulae are known from the triangle geometry

$$r_a + r_b + r_c = 4R + r,\ r_b r_c + r_c r_a + r_a r_b = p^2,\ r_a r_b r_c = p^2 r.$$

Denote $\sigma_1 = r_a + r_b + r_c$, $\sigma_2 = r_b r_c + r_c r_a + r_a r_b$, $\sigma_3 = r_a r_b r_c$. Therefore, $\sigma_1 = 4R + r$, $\sigma_2 = p^2$, $\sigma_3 = p^2 r$. We have from the equality which is implemented in the table, that

$$r_a^3 r_b^2 + r_a^2 r_b^3 + r_b^3 r_c^2 + r_b^2 r_c^3 + r_c^3 r_a^2 + r_c^2 r_a^3 = \sigma_1 \sigma_2^2 - 2\sigma_1^2 \sigma_3 - \sigma_2 \sigma_3$$

and $r_a^3 r_b^2 + r_a^2 r_b^3 + r_b^3 r_c^2 + r_b^2 r_c^3 + r_c^3 r_a^2 + r_c^2 r_a^3 = 2p^2\left(2Rp^2 - 16R^2 r - 8Rr^2 - r^3\right)$.

Using the well-known inequations $p^2 \le 4R^2 + 4Rr + 3r^2$, $p^2 \ge 16Rr - 5r^2$, $R \ge 2r$ and executing some transformations we obtain the needed inequalities.

Problem 3. Prove that the following inequation is verified for all triplets of positive numbers a, b and c

$$\frac{(b+c-a)^2}{(b+c)^2 + a^2} + \frac{(c+a-b)^2}{(c+a)^2 + b^2} + \frac{(a+b-c)^2}{(a+b)^2 + c^2} \ge \frac{3}{5}.$$

Solution. The inequation under consideration is equivalent to the next one

$$\frac{2\left[(b+c)^2 + a^2\right] - (a+b+c)^2}{(b+c)^2 + a^2} + \frac{2\left[(c+a)^2 + b^2\right] - (a+b+c)^2}{(c+a)^2 + b^2} +$$
$$+ \frac{2\left[(a+b)^2 + c^2\right] - (a+b+c)^2}{(a+b)^2 + c^2} \ge \frac{3}{5}.$$

We obtain from the last that

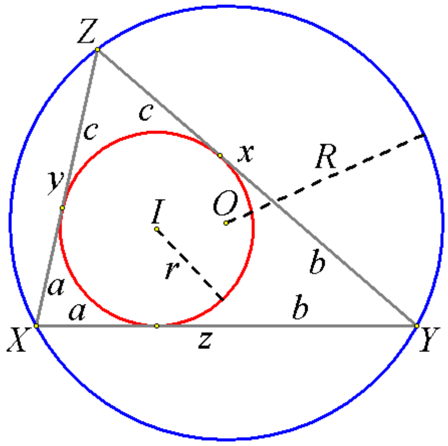
$$\frac{(a+b+c)^2}{(b+c)^2 + a^2} + \frac{(a+b+c)^2}{(c+a)^2 + b^2} + \frac{(a+b+c)^2}{(a+b)^2 + c^2} \le \frac{27}{5}.$$

Put $\sigma_1 = a + b + c$, $\sigma_2 = ab + bc + ca$, $\sigma_3 = abc$. After some not complicated transformations we obtain

$$\frac{\sigma_1^2\left(3\sigma_1^4 - 8\sigma_1^2\sigma_2 + 4\sigma_1\sigma_3 + 4\sigma_2^2\right)}{\sigma_1^6 - 4\sigma_1^4\sigma_2 + 4\sigma_1^3\sigma_3 + 4\sigma_1^2\sigma_2^2 - 8\sigma_1\sigma_2\sigma_3 + 8\sigma_3^2} \le \frac{27}{5}.$$

The denominator in the right-hand side of this inequation is positive. It follows that the inequation is equivalent to the next one

$$3\sigma_1^6 - 17\sigma_1^4\sigma_2 + 22\sigma_1^3\sigma_3 + 22\sigma_1^2\sigma_2^2 - 54\sigma_1\sigma_2\sigma_3 + 54\sigma_3^2 \ge 0.$$



Since the numbers a, b and c are positive, then the numbers $x=b+c$, $y=c+a$ and $z=a+b$ could be considered as side lengths of triangle XYZ. If $p=\frac{x+y+z}{2}$, then $a=p-x$, $b=p-y$ and $c=p-z$. Let R and r be the radii of the circumcircle and the incircle of ΔXYZ, respectively. Since $x+y+z=2p$, $yz+zx+xy=p^2+r^2+4Rr$ and $xyz=4pRr$, then $\sigma_1=p$, $\sigma_2=r^2+4Rr$ and $\sigma_3=pr^2$. We obtain after substituting in the last inequation, that

$$p^2\left[3p^4-(68R-5r)rp^2\right]+2\left(176R^2-20Rr+11r^2\right)r^2\geq 0.$$

This inequation is equivalent to the following

$$\left(6p^2-68Rr+5r^2-r\sqrt{400R^2-200Rr-239r^2}\right)\times$$
$$\times\left(6p^2-68Rr+5r^2+r\sqrt{400R^2-200Rr-239r^2}\right)\geq 0.$$

Using the inequations $p^2\geq 16Rr-5r^2$ and $R\geq 2r$, we obtain

$$6p^2-68Rr+5r^2\geq 6\left(16Rr-5r^2\right)-68Rr+5r^2=(28R-25r)r\geq(28.2r-25r)r=31r^2>0.$$

Consequently, $6p^2-68Rr+5r^2+r\sqrt{400R^2-200Rr-239r^2}>0$.

Since $6p^2-68Rr+5r^2\geq(28R-25r)r>0$, then the inequation

$$6p^2-68Rr+5r^2-r\sqrt{400R^2-200Rr-239r^2}\geq 0$$

is equivalent to $(28R-25r)r\geq 400R^2-200Rr-239r^2$. It could be noticed after some known transformations, that the last is equivalent to $(8R-9r)(R-2r)\geq 0$. Since $R\geq 2r$, then the last inequation is satisfied always. This ends the proof of the inequation

$$6p^2-68Rr+5r^2-r\sqrt{400R^2-200Rr-239r^2}\geq 0.$$

The results show, that the inequation under consideration is satisfied for all positive real numbers a, b and c. Equality appears in the last inequation in the case of equilateral ΔXYZ. Thus, equality appears in the initial inequation in the case $a=b=c$ only.

Meanwhile, we obtain the inequation

$$\frac{p^2}{x^2+(p-x)^2}+\frac{p^2}{y^2+(p-y)^2}+\frac{p^2}{z^2+(p-z)^2}\leq\frac{27}{5}$$

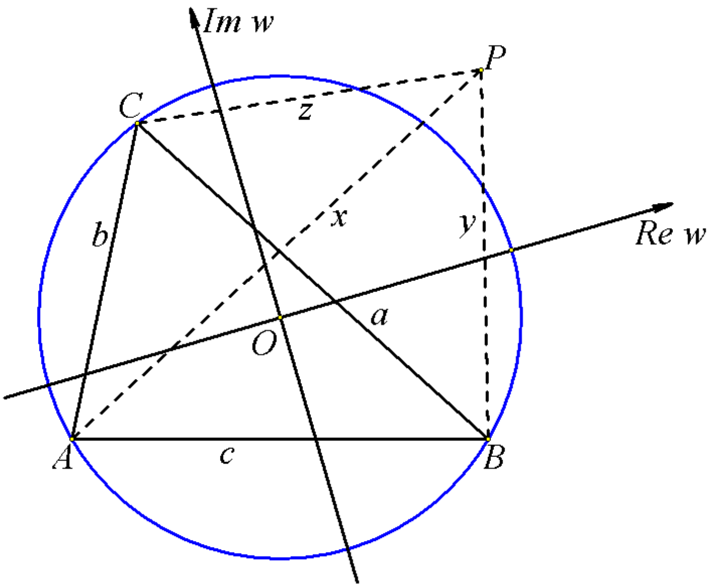
for all ΔXYZ.

Problem 4. Let x, y and z be the distances from arbitrary point P in the plane of a given triangle ABC to the vertices A, B and C, respectively. If $|AB|=c$, $|BC|=a$, $|CA|=b$, prove the inequation

$$a^2x^4+b^2y^4+c^2z^4+\left(a^2-b^2-c^2\right)\left(a^2x^2+y^2z^2\right)+\left(b^2-a^2-c^2\right)\left(b^2y^2+z^2x^2\right)+$$
$$+\left(c^2-a^2-b^2\right)\left(c^2z^2+x^2y^2\right)+a^2b^2c^2=0.$$

Solution. Consider ΔABC in the complex plane with respect to Gauss coordinate system, for which the circumcircle of ΔABC is the unit one. If the affixes of the vertices A, B and C are a_0, b_0 and c_0, respectively, then

$$a_0\overline{a}_0=b\overline{b}_0=c_0\overline{c}_0=1,\ a^2=-\frac{(b_0-c_0)^2}{b_0c_0},\ b^2=-\frac{(c_0-a_0)^2}{c_0a_0},\ c^2=-\frac{(a_0-b_0)^2}{a_0b_0}.$$



If p is the affix of the point P, then $x^2=(p-a_0)(\overline{p}-\overline{a}_0)$, $z^2=(p-c_0)(\overline{p}-\overline{c}_0)$ and $x^2=(p-a_0)(\overline{p}-\overline{a}_0)$.

Introduce the denotations

$$M=a^2x^4+b^2y^4+c^2z^4,$$

$$N=(a^2-b^2-c^2)(a^2x^2+y^2z^2)+(b^2-a^2-c^2)(b^2y^2+z^2x^2)+(c^2-a^2-b^2)(c^2z^2+x^2y^2),$$

$$\sigma_1=a_0+b_0+c_0,\ \sigma_2=b_0c_0+c_0a_0+a_0b_0,\ \sigma_3=a_0b_0c_0.$$

After substituting the values of a, b, c, x, y and z from above and after some transformations we obtain

$$M=-\frac{1}{\sigma_3^2}\Big[\sigma_3(\sigma_1\sigma_2-9\sigma_3)p\overline{p}(p\overline{p}+4)-4\sigma_3(\sigma_2^2-3\sigma_1\sigma_3)(p\overline{p}+1)(p+\overline{p})+$$

$$+(\sigma_1^2\sigma_2-4\sigma_2^2+3\sigma_1\sigma_3)(p^2+\sigma_3\overline{p}^2)+\sigma_3(\sigma_1\sigma_2-9\sigma_3)\Big],$$

$$N=-\frac{1}{\sigma_3^2}\Big[\sigma_3(\sigma_1\sigma_2-9\sigma_3)p\overline{p}(p\overline{p}+4)-4\sigma_3(\sigma_2^2-3\sigma_1\sigma_3)(p\overline{p}+1)(p+\overline{p})+$$

$$+(\sigma_1^2\sigma_2-4\sigma_2^2+3\sigma_1\sigma_3)(p^2+\sigma_3\overline{p}^2)+\sigma_1^2\sigma_2^2-4\sigma_1^3\sigma_3+19\sigma_1\sigma_2\sigma_3-4\sigma_2^3-36\sigma_3^2\Big].$$

It follows that

$$M-N=-\frac{\sigma_1^2\sigma_2^2-4\sigma_1^3\sigma_3+18\sigma_1\sigma_2\sigma_3-4\sigma_2^3-27\sigma_3^2}{\sigma_3^2}=$$

$$=\left[\frac{(b_0-c_0)(c_0-a_0)(a_0-b_0)}{a_0b_0c_0}\right]^2=-a^2b^2c^2.$$

This ends the proof.

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